JMO Solutions

Section A

A1 10° At 4 p.m. the minute hands points to the '4' but the hour hand has moved from the '4' in the preceding 20 minutes. The hour hand rotates through 360° in 12 hours, which means that it rotates 30° in one hour and therefore 10° in 20 minutes.

A2 4 ways The four ways are: 3 + 47; 7 + 43; 13 + 37; 19 + 31.

A3 132

$$\frac{{}^{8}/_{6}}{99} = \frac{8}{6} \times \frac{1}{99} = \frac{4}{297}; \qquad \frac{8}{{}^{6}/_{99}} = \frac{8}{1} \times \frac{99}{6} = 132.$$

The sum of the fractions is therefore $132\frac{4}{297}$.

A4 9 If increasing the cost of an adult's ticket by £1 and reducing the cost of a child's ticket by £1 leaves the total cost unchanged, then the number of adults must equal the number of children. If there are n adults and n children, then $7n + 5n = 108$ so $12n = 108$ and $n = 9$.

A5 5 There are 24 possible arrangements of the letters U, K, M and T ($1 \times 2 \times 3 \times 4$) and there are 120 possible arrangements of the letters U, K, J, M and O ($1 \times 2 \times 3 \times 4 \times 5$). Therefore $\frac{p}{q} = \frac{120}{24} = 5$.

A6 46 If you consider the cube to be made up of 4 horizontal layers, each containing 16 small cubes before the holes are made, then the number of cubes remaining in these layers are 14, 6, 12 and 14 respectively.

A7£9 6s 8d 8d × 7 = 56d = 4s 8d; 6s × 7 = 42s; 42s + 4s (carry) = 46s = £2 6s; £1 × 7 = £7; £7 + £2 (carry) = £9. [Note that £1 6s 8d × 7 = £9 6s 8d. This is not a coincidence as 6s 8d was exactly one third of £1.]

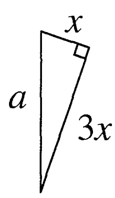
A8 $\frac{\pi}{8}$ Let the length of the side of the cube be x. Therefore the surface area of the cube is $6x^2$. The area of each shaded quarter-circle is $\pi x^2/4$ and therefore the fraction of the surface area which is shaded is

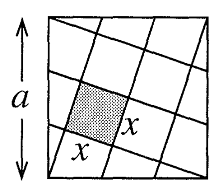
$$\frac{\left(\frac{3\pi x^2}{4}\right)}{6x^2} = \frac{3\pi}{24} = \frac{\pi}{8}.$$

A9 $n = 4$

$$\begin{aligned} 0.006 + 0.04n &= (n + 6) \times 0.0166 \\ 60 + 400n &= 166(n + 6) \\ 60 + 400n &= 166n + 996 \\ 234n &= 936 \\ n &= 4 \end{aligned}$$

A10 $\frac{1}{10}$ Applying Pythagoras' Theorem to the triangle shown gives $a^2 = x^2 + (3x)^2$ i.e. $a^2 = 10x^2$. But the areas of the small square and the large square are x^2 and a^2 and so $x^2 = \frac{1}{10}a^2$ gives the answer.





Section B

B1 If N is a multiple of 15 then it must be both a multiple of 3 and a multiple of 5. To be a multiple of 3, the sum of its digits must also be a multiple of 3 and to be a multiple of 5, N's unit digit must be 5 or 0.

If N's unit digit is 0 then its first digit must be 3, 6 or 9 in order for the sum of the digits to be a multiple of 3. Thus 3120, 6120 and 9120 are possible values of N.

If N's unit digit is 5 then its first digit must be 1, 4 or 7 in order for the sum of the digits to be a multiple of 3. The only other possible values of N, therefore, are 1125, 4125 and 7125.

B2 When $\angle AOB = 42°$, $\angle AOP = \angle POQ = \angle QOB = 14°$. $\angle BOC = 180° - 42° = 138°$ and therefore $\angle BOR = \angle ROS = \angle SOC = 46°$.

Hence $\angle QOR = 14° + 46° = 60°$ and $\angle POS = 14° + 14° + 46° + 46° = 120°$.

When $\angle AOB = x°$, $\angle BOC = 180° - x°$ and therefore $\angle BOR = \angle ROS = \angle SOC = \left(60 - \frac{1}{3}x\right)°$.

Therefore $\angle QOR = \angle QOB + \angle BOR = \frac{1}{3}x° + \left(60 - \frac{1}{3}x\right)° = 60°$.

B3 The total number of points won was 35. The information given makes it impossible for there to have been 35 events worth 1 point each, nor could there have been just 1 event worth 35 points. There must, therefore, have been 5 events worth 7 points each, or 7 events worth 5 points each. Each event must have been worth at least 6 points (3 + 2 + 1), however, and therefore it may be deduced that there were 5 events with each event being worth 7 points: 4 points for 1st place, 2 points for 2nd place and 1 point for 3rd place.

It may now be deduced that Alice's points came from four 1st places and one 2nd place whilst the March Hare won one event, the Sack Race, and came 3rd in the other four. The Mock Turtle came 2nd in four events and 3rd in the Sack Race since that must have been the event in which Alice finished 2nd. Last place in the ‘Egg and Spoon’ race, therefore, was filled by the March Hare.

B4 Let O be the centre of both regular hexagons. Triangle POQ is equilateral and therefore the length of each side of $PQRSTU$ is equal to the length of PO.

Considering triangle OAB: $PO^2 + 1^2 = 2^2 \Rightarrow PO = \sqrt{3}$. POQ is therefore an equilateral triangle of side $\sqrt{3}$ and its height, h, may be found using Pythagoras' Theorem or by using similar triangles. By Pythagoras:

$$h^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = (\sqrt{3})^2 \Rightarrow h^2 + \frac{3}{4} = 3 \Rightarrow h^2 = \frac{9}{4} \Rightarrow h = \frac{3}{2}.$$

Using similar triangles (ΔPOQ, ΔAOB): $\frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow h = \frac{3}{2}$.

The area of triangle POQ is therefore $\frac{1}{2} \times \sqrt{3} \times \frac{3}{2} = \frac{3\sqrt{3}}{4}$ and the area of $PQRSTU$ is this multiplied by 6 = $6 \times \frac{3\sqrt{3}}{4} = \frac{18\sqrt{3}}{4} = \frac{9\sqrt{3}}{2}$.

[A shorter method, having found the length of PO, is to use the fact that in similar figures, ratio of areas = (ratio of side lengths)^2.

Therefore: area of $PQRSTU$ = $\left(\frac{\sqrt{3}}{2}\right)^2 \times$ area of $ABCDEF$ = $\frac{3}{4} \times 6 \times \frac{1}{2} \times 2 \times \sqrt{3} = \frac{9\sqrt{3}}{2}$.]

B5 (a) Consider the two-digit number whose first digit is a and whose second digit is b: the value of the number is $10a + b$. When its digits are reversed, the value becomes $10b + a$. As the number is increased by 75% when the digits are reversed

$$10b + a = 1.75 \times (10a + b) \Rightarrow 4(10b + a) = 7(10a + b)$$

$$\Rightarrow 40b + 4a = 70a + 7b \Rightarrow 33b = 66a \text{ and so } b = 2a.$$

As a and b are both digits, the only solutions for (a, b) are (1, 2), (2, 4), (3, 6), (4, 8) giving 12, 24, 36 and 48 as the only two-digit numbers with the required property.

(b) Letting the three digits of the original number be a, b and c respectively, we have the condition that:

$$100c + 10b + a = 1.75 \times (100a + 10b + c).$$

Therefore $4(100c + 10b + a) = 7(100a + 10b + c)$

$$400c + 40b + 4a = 700a + 70b + 7c$$

$$393c = 696a + 30b$$

$$131c = 232a + 10b.$$

Notice that the right hand side will be even, whatever the values of a and b, and therefore c must be even.

If $c = 2$: $262 = 232 + 10b$ and this is satisfied only by $a = 1$ and $b = 3$ (since a and b are both single digits).

If $c = 4$: $524 = 232a + 10b$ and this is satisfied only by $a = 2$ and $b = 6$.

If $c = 6$: $786 = 232a + 10b$ and this is satisfied only by $a = 3$ and $b = 9$.

If $c = 8$: $1048 = 232a + 10b$ and there are no single digit solutions.

Therefore 132, 264 and 396 are the only three-digit numbers with the required property.

B6 (Note: in this solution s_1 refers to the first square, s_2 to the second square and so on.)

(a) X wins by placing his first counter on squares s_2 and s_3. Y is then unable to place a counter on the board.

(b) Y wins on a 5×1 board. If X places his counter on s_1 and s_2 then Y wins whether he places his counter on s_3 and s_4 or on s_4 and s_5; if X places his counter on s_2 and s_3 then Y places his on s_4 and s_5 and wins. X's other two possible first moves are simply reflections of those already considered and therefore Y must win on a 5×1 board.

X will win on a 7×1 board. We have seen that the second person to play wins on a 5×1 board. Therefore if X places his first counter on s_1 and s_2 then the game becomes the same as a 5×1 game with Y playing first and therefore X winning.

(c) X will win on a 6×1 board by placing his first counter on s_3 and s_4. If Y goes on s_1 and s_2 then X goes on s_5 and s_6 or vice versa.

(d) X will also win on a 8×1 board. He places his first counter on s_4 and s_5 and then places his next counter so that it is a reflection of Y's counter in the line of symmetry through the centre of the board e.g. if Y goes on s_6 and s_7 then X goes on s_2 and s_3.

(By using this tactic of placing his first counter on the two centre squares and then simply reflecting Y's counters, X will always win when the number of squares on the board is even.)